Propagation of a cylindrically symmetric wave through an optical system

1974 J. Phys. A: Math. Nucl. Gen. 7 519

(http://iopscience.iop.org/0301-0015/7/4/015)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 4, 1974. Printed in Great Britain. © 1974

Propagation of a cylindrically symmetric wave through an optical system

Eero Byckling
Department of Physics, University of Jyväskylä, Jyväskylä, Finland

Received 30 May 1973

Abstract. The propagation of an electromagnetic wave through an optical system is studied for the case when the wave has rotational symmetry with respect to the symmetry axis of the system. Spherical surfaces are assumed. Using a method previously developed for the refraction at a single surface, which makes use of expansions in spherical harmonics, we now solve the problem of combining such expansions to treat a system of arbitrary complexity.

1. Introduction

In a previous paper (Byckling 1974, to be referred to as I) we found that it is possible to calculate the intensity distribution at the image of a point object, if the optical system is a single spherical surface separating two media of different refractive indices. We used there some methods of modern theoretical physics, in particular eigenfunction expansions and complex integration. An essential ingredient is the principle of stationary phase. This is used to evaluate spherical Bessel functions $j_l(x)$, $h_l(x)$ and Legendre functions $P_l^m(\cos\theta)$ for large order. Even more importantly, it allows the calculation of complicated integrals and leads to simple results. The physical content of this principle is phase coherence, only regions in which the wave components have the same phase can contribute to the intensity, others can be neglected.

In this paper we solve the problem of transforming an expansion in spherical harmonics from one coordinate system to another. We then use this solution and the results of paper I to calculate the amplitude distribution of a wave that has passed through a system of n spherical boundaries. Cylindrical symmetry is assumed so that only an axial object point can be treated. Off-axis object points will be the subject of a subsequent publication.

We note in passing that the method used here could be formulated in terms of the theory of the representations of the three-dimensional group of rotations and translations. This, in addition to the principle of stationary phase, is the reason behind the simplicity of some of the results. In this paper, however, we only make use of techniques taken from calculus.

2. Transformation of an axially symmetric field at the boundary of two media

Assume that there is a source of radiation on the z axis and the amplitude of the source is given by the complex function $g(z)$. We further assume that the source is of finite extent so that $g(z) = 0$ for $|z| > \hat{z}$. Then assuming that the resulting electromagnetic field $A(\boldsymbol{r})$ is symmetric in rotations around the z axis, one has according to I that

$$A(\boldsymbol{r}) = \int \mathrm{d}z\, g(z) G(\boldsymbol{z}, \boldsymbol{r}). \tag{2.1}$$

The propagator $G(\boldsymbol{z}, \boldsymbol{r})$ can be written in terms of the angle θ between $\boldsymbol{z}$ and $\boldsymbol{r}$ and the lengths z and r. One must distinguish two cases (van Bladel 1964)

$$G(\boldsymbol{z}, \boldsymbol{r}) = -ink \sum_l (2l+1) P_l(\cos\theta) j_l(nkz) h_l^{(2)}(nkr), \qquad z < r \tag{2.2}$$

$$G(\boldsymbol{z}, \boldsymbol{r}) = -ink \sum_l (2l+1) P_l(\cos\theta) h_l^{(2)}(nkz) j_l(nkr), \qquad z > r \tag{2.3}$$

In the case $r = z$ these are identical, and they then have a singularity at $\cos\theta = 1$. If the source is completely inside the sphere $\boldsymbol{r} = r$, the amplitude can be written

$$A(\boldsymbol{r}) = \sum_l \frac{2l+1}{4\pi} B_l P_l(\cos\theta) h_l^{(2)}(nkr) \tag{2.4}$$

$$B_l = -ink[4\pi(2l+1)]^{1/2} \int \mathrm{d}z\, j_l(nkz) g(z). \tag{2.5}$$

If the source is outside the sphere $\boldsymbol{r} = r$, one has

$$A(\boldsymbol{r}) = \sum_l \left(\frac{2l+1}{4\pi}\right)^{1/2} C_l P_l(\cos\theta) j_l(nkr) \tag{2.6}$$

$$C_l = -ink[4\pi(2l+1)]^{1/2} \int \mathrm{d}z\, h_l^{(2)}(nkr) g(z). \tag{2.7}$$

The wavefield in the second medium was, in equation (2.35) in I, given for a point source. An extended source $g(z)$ will only modify the calculation to the extent that one must make the replacement

$$h_l^{(2)}(nkt) \to \int \mathrm{d}z\, h_l^{(2)}(nkz) g(z) = \frac{C_l}{-ink[4\pi(2l+1)]^{1/2}}. \tag{2.8}$$

Then equations (2.35) and (A.7) in I give

$$A(\boldsymbol{t}') = \left(\frac{8}{\pi}\right)^{1/2} in'k \sum_{\check{l} < l' < \hat{l}} l'^{1/2} j_{l'}(nkr) j_{l'}(n'kr) h_{l'}^{(2)}(n'kt') C_{l'} P_{l'}(\cos\alpha'). \tag{2.9}$$

The limits $\check{l}, \hat{l}$ are due to the constraint $|\chi| = |\partial\phi/\partial v| \leqslant \hat{\theta}$ (see (2.20) and (2.33) in I).

Comparing with (2.6) we see that $A(\boldsymbol{t}')$ is of the form

$$A(\boldsymbol{t}') = \sum_{l'} \left(\frac{2l'+1}{4\pi}\right)^{1/2} C'_{l'} P_{l'}(\cos\alpha') h_{l'}^{(2)}(n'kt') \tag{2.10}$$

with

$$C'_{l'} = \begin{cases} 0 & l' < \check{l} \\ 4in'kj_{l'}(nkr) j_{l'}(n'kr) C_{l'} & \check{l} < l' < \hat{l} \\ 0 & l' > \hat{l}. \end{cases} \tag{2.11}$$

The coefficients C_l thus have an extremely simple transformation formula in the refraction. Outside the interval $\check{l}, \hat{l}$ they are put to zero, and inside it they are multiplied by a simple function of l. For a point source at $z = t$ the C_l are

$$C_l = -ink[4\pi(2l+1)]^{1/2} h_l^{(2)}(nkt) \tag{2.12}$$

and for a point image at t' one has a similar expression. The interesting property is the deviation from the ideal imaging process. Thus one can define

$$D_l = \frac{C_l}{-ink[4\pi(2l+1)]^{1/2} h_l^{(2)}(nkt)} = \frac{\int \mathrm{d}z\, h_l^{(2)}(nkz) g(z)}{h_l^{(2)}(nkt)}. \tag{2.13}$$

Then (2.10) becomes

$$A(t') = \sum_{l_1 < l' < l_2} 8nn'k^2 l' j_{l'}(nkr) j_{l'}(n'kr) h_{l'}^{(2)}(nkt) h_{l'}^{(2)}(n'kt') D_{l'} P_{l'}(\cos\alpha'). \tag{2.14}$$

If now the phase of $j_{l'} j_{l'} h_{l'}^{(2)} h_{l'}^{(2)} D_{l'}$ in (2.14) is nearly independent of l', $A(t')$ at $\cos\alpha' = 1$ is large.

We now choose a fixed value of t' (for a given t) and study the phase in (2.14). There are two choices of t' which appear natural. One can take the gaussian image point determined by

$$\frac{1}{nr} - \frac{1}{n'r} + \frac{1}{nt} + \frac{1}{n't'} = 0. \tag{2.15}$$

Then the phase is stationary at $l = 0$.

The second possibility is to choose t' at the maximum intensity. This value t' is obtained by finding the minimum of $B = B(l_0)$, equation (I.2.30), where $l_0 = l_0(t')$ is the stationary value given by $\partial\phi/\partial l = 0$. This second possibility probably leads to easier numerical calculations when aberrations are large. We discuss here only the first alternative, which leads to simpler formulae and becomes equivalent to the second when the diffraction limit is approached.

We write (2.14) as

$$A(t') = \sum_{\check{l} < l < \hat{l}} F_l(t, t') D_l P_l(\cos\alpha'). \tag{2.16}$$

At large l the factor F_l is

$$F_l(t, t') = \mathrm{e}^{\mathrm{i}\phi_l} \frac{1}{r(tt')^{1/2}} (n^2k^2r^2 - l^2)^{-1/4} (n'^2k^2r^2 - l^2)^{-1/4} (n^2k^2t^2 - l^2)^{-1/4} \times (n'^2k^2t'^2 - l^2)^{-1/4} \tag{2.17}$$

$$\phi_l = \pm\Phi_l \pm \Phi_l' - \Psi_l - \Psi_l'. \tag{2.18}$$

The phase factors Φ, Ψ are

$$\Phi_l = (n^2k^2r^2 - l^2)^{1/2} - l\cos^{-1}\left(\frac{l}{nkr}\right) - \frac{\pi}{4} \tag{2.19}$$

$$\Psi_l = (n^2k^2t^2 - l^2)^{1/2} - l\cos^{-1}\left(\frac{l}{nkt}\right) - \frac{\pi}{4} \tag{2.20}$$

and Φ_l', Ψ'_l are obtained through the replacement

$$n \to n' \qquad t \to t'. \tag{2.21}$$

For $l = 0$ the phase is

$$\phi_0 = \pm(nkr - \tfrac{1}{4}\pi) \pm (n'kr - \tfrac{1}{4}\pi) - (nkt - \tfrac{1}{4}\pi) - (n'kt' - \tfrac{1}{4}\pi). \tag{2.22}$$

This is an uninteresting change of phase related to the propagation from t to t' along the z axis. The important quantity is the variation of the phase, $\phi_l - \phi_0$, with l. Because of (2.15) the phase is stationary at $l = 0$, and the image quality is dependent on how constant the phase $\phi_l - \phi_0$ is when l increases from $\check{l}$ to $\hat{l}$.

3. Transformation of an expansion from one coordinate system to another

To be able to treat successive refractions, we need to study the change of the expansion in spherical harmonics when the field is expressed with respect to a second coordinate system. We discuss here only cylindrically symmetric wavefields.

Let S and S′ be two coordinate systems related by

$$x' = x, \qquad y' = y, \qquad z' = z - d. \tag{3.1}$$

Assume that the wavefield is written as the expansion (2.4). Then it can be regarded as due to a source $g(z)$ in the interval $-r < z < r$. Using the formula

$$\int_{-\infty}^{\infty} \mathrm{d}x\, j_l(x) j_{l'}(x) = \frac{\pi}{2l+1} \delta_{ll'} \tag{3.2}$$

we can show that the source has the expansion

$$g(z) = \frac{\mathrm{i}}{2\pi^{3/2}} \sum_{l=0}^{\infty} (2l+1)^{1/2} B_l j_l(nkz). \tag{3.3}$$

Now we ask what are the expansion coefficients in coordinate system S′ for the field created by the source (3.3)?

We first consider such a point $\boldsymbol{r}'$ in S′ that the source will be totally inside the sphere $r' = r'$. According to (2.4), (2.5), the field is

$$A(\boldsymbol{r}') = \sum_{l'} \left(\frac{2l'+1}{4\pi} \right)^{1/2} B'_{l'} P_{l'}(\cos\theta') h_{l'}^{(2)}(nkr') \tag{3.4}$$

with

$$\begin{aligned} B'_{l'} &= -\mathrm{i}nk[4\pi(2l'+1)]^{1/2} \int \mathrm{d}z'\, j_{l'}(nkz') g(z') \\ &= \frac{nk}{\pi} \sum_{l=0}^{\infty} (2l+1)^{1/2} (2l'+1)^{1/2} \int \mathrm{d}z'\, j_l(nkz' + nkd) j_{l'}(nkz') B_l. \end{aligned} \tag{3.5}$$

From appendix 1, equation (A.7) we then get

$$B'_{l'} = \sum_l (j_{|l-l'|}(nkd) + j_{l+l'}(nkd)) B_l. \tag{3.6}$$

If the point $z' = -d$ is outside the sphere $\boldsymbol{r}' = r'$, ie if $r' < d$, then the relevant expansion is

$$A(\boldsymbol{r}') = \sum_{l'} \left(\frac{2l'+1}{4\pi} \right)^{1/2} C'_{l'} P_{l'}(\cos\chi') j_{l'}(nkr') \tag{3.7}$$

$$C'_{l'} = \sum_l (h_{|l-l'|}^{(2)}(nkd) + h_{l+l'}^{(2)}(nkd)) B_l. \tag{3.8}$$

Finally, if the expansion over l is of the type (2.6), the two cases $r' > d$ and $r' < d$ result in

$$B'_{l'} = \sum_l (h^{(2)}_{|l-l'|}(nkd) + h^{(2)}_{l+l'}(nkd))C_l \tag{3.9}$$

$$C'_{l'} = \sum_l h^{(2)}_{|l-l'|}(nkd)C_l. \tag{3.10}$$

In summary, when an expansion in S in transformed into an expansion in S′, the coefficients A_l are transformed according to

$$A'_{l'} = \sum_l K_{l'l}(nkd)A_l \tag{3.11}$$

where the kernel $K_{l'l}$ can be read from (3.6), (3.8)–(3.10), depending on the type of the two expansions. Actually, we shall only need the parts propagating to the positive z direction (equation (3.10)).

4. Propagation of an axially symmetric field through several surfaces

We now consider a set of n spherical surfaces $\sigma_1, \ldots, \sigma_n$. The notation is taken as in figure 1. The distance d_i is positive if O_i is to the right of O_{i-1}, otherwise it is negative. The radius r_i is positive if the surface σ_i intersects the z axis to the left of O_i, otherwise it is negative, $-|r_i|$.

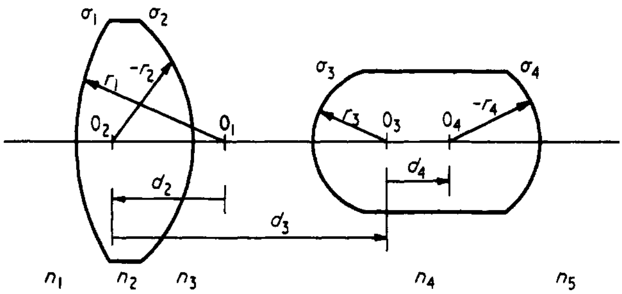

Figure 1. Notation and sign conventions.

One now fixes the object point P_0 and calculates the positions of the successive paraxial image points $P_1, \ldots, P_n$ from the formulae

$$\frac{1}{n_i r_i} + \frac{1}{n_i t_i} = \frac{1}{n_{i+1} r_i} + \frac{1}{n_{i+1} t'_i} \tag{4.1}$$

$$t_{i+1} = t'_i - d_{i+1}. \tag{4.2}$$

As seen from figure 2, the distances t_i, t'_i are positive when going to the positive z direction, otherwise they are negative. The object distance from O_1 is t_1 and image distance from O_n is t'_n.

The field between surfaces σ_i and σ_{i+1} is written as an expansion with respect to point O_i and also as an expansion with respect to O_{i+1}. In writing these we have to elaborate slightly the sign conventions. The common convention for the sign of r_i,

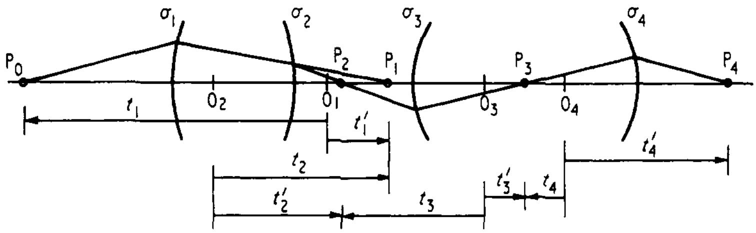

Figure 2. The successive paraxial image points $P_0, P_1, \ldots, P_n$ and the definitions of t_i, t_i'.

defined above, is the opposite of what would be convenient here, but irrespective of that we shall adhere to it. Then the field between σ_i and σ_{i+1} propagating to the right can be written in the forms

$$A^{(i)}(\boldsymbol{r}_i) = \sum_l \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta_i) B_l^{(i)} h_l^{(1)}(n_{i+1} k r_i) \tag{4.3}$$

$$A^{(i)}(\boldsymbol{r}_{i+1}) = \sum_l \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta_{i+1}) \bar{B}_l^{(i)} h_l^{(1)}(n_{i+1} k r_{i+1}). \tag{4.4}$$

In (4.3) and (4.4) the phases depend on r_i and r_{i+1} so that moving the point to the positive z direction will give a factor $e^{-i\Delta}$. This is seen for both signs of r from the equations

$$r < 0 \Rightarrow h_l^{(1)}(r-\Delta) = h_l^{(1)}[-(|r|+\Delta)] = (-1)^l h_l^{(2)}(|r|+\Delta) \simeq e^{-i\Delta} \tag{4.5}$$

$$r > 0 \Rightarrow h_l^{(1)}(r-\Delta) \simeq e^{-i\Delta}. \tag{4.6}$$

We can now define the vectors

$$\boldsymbol{B}^{(i)} = (B_0^{(i)}, B_1^{(i)}, \ldots) \tag{4.7}$$

$$\bar{\boldsymbol{B}}^{(i)} = (\bar{B}_0^{(i)}, \bar{B}_1^{(i)}, \ldots). \tag{4.8}$$

Then the sequence

$$\bar{\boldsymbol{B}}^{(0)}, \boldsymbol{B}^{(1)}, \bar{\boldsymbol{B}}^{(1)}, \ldots, \boldsymbol{B}^{(n-1)}, \bar{\boldsymbol{B}}^{(n-1)}, \boldsymbol{B}^{(n)} \tag{4.9}$$

gives the successive wavefields and it can be calculated by successive integral transforms using the results of §§ 2 and 3.

The refraction at surface σ_i causes a transformation of the expansion coefficients:

$$\boldsymbol{B}^{(i+1)} = \boldsymbol{K}^{(i)} \bar{\boldsymbol{B}}^{(i)}. \tag{4.10}$$

The transformation matrix $\boldsymbol{K}^{(i)}$ is obtained using (2.11) and the equation

$$j_l(x) = \tfrac{1}{2} h_l^{(1)}(x) + \tfrac{1}{2} h_l^{(2)}(x), \tag{4.11}$$

and taking only the part propagating in the positive z direction. Then the matrix $\boldsymbol{K}^{(i)}$ is seen to be diagonal with elements

$$K_{ll'}^{(i)} = 0 \qquad l \neq l'$$

$$K_{ll}^{(i)} = \begin{cases} 0 & l < \check{l} \\ i n_{i+1} k h_l^{(1)}(n_i k r_i) h_l^{(2)}(n_{i+1} k r_i) & \check{l} < l < \hat{l} \\ 0 & \hat{l} < l. \end{cases} \tag{4.12}$$

The values $\check{l}$ and $\hat{l}$ are the end points of the l interval which satisfies the condition

$$\hat{\theta}_i \geqslant \left| \pi - \tfrac{1}{2}\left[\cos^{-1}\left(\frac{l}{n_i k r_i}\right) + \cos^{-1}\left(\frac{l}{n_i k t_i}\right) + \cos^{-1}\left(\frac{l}{n_{i+1} k r_i}\right) + \cos^{-1}\left(\frac{l}{n_{i+1} k t'_i}\right)\right]\right|. \tag{4.13}$$

Here $\hat{\theta}_i$ is the angular aperature of surface σ_i as seen from point O_i. In this discussion t'_i is the value given by (4.1) and then always $\check{l} = 0$.

The field that propagates in the region between σ_i and σ_{i+1} has the expansion (4.3) with respect to O_i and (4.4) with respect to O_{i+1}. The coefficient vectors $\boldsymbol{B}^{(i)}$ and $\bar{\boldsymbol{B}}^{(i)}$ are related by a matrix $\boldsymbol{L}^{(i)}$:

$$\bar{\boldsymbol{B}}^{(i)} = \boldsymbol{L}^{(i)}\boldsymbol{B}^{(i)}. \tag{4.14}$$

According to § 3 the kernel $\boldsymbol{L}^{(i)}$ has the elements

$$L^{(i)}_{l'l} = h^{(2)}_{|l-l'|}(nkd). \tag{4.15}$$

If d is negative, this is actually $h^{(1)}$:

$$h^{(2)}_l(-x) = (-1)^l h^{(1)}_l(x). \tag{4.16}$$

The coefficients $\bar{B}^{(0)}_l$ of an initial field created by a point object are, according to (2.7), equal to

$$\bar{B}^{(0)}_l = -in_1k[4\pi(2l+1)]^{1/2}h^{(1)}_l(n_1kt_1). \tag{4.17}$$

If the object is at $-\infty$, then we use the well known formula

$$e^{iz\cos\theta} = \sum_{l=0}^{\infty} (2l+1)i^l j_l(z) P_l(\cos\theta) \tag{4.18}$$

to obtain

$$\bar{B}^{(0)}_l = [4\pi(2l+1)]^{1/2}(-i)^l. \tag{4.19}$$

The amplitude in the image is given by the coefficients $B^{(n)}_l$ and at a point t', α' we have from (2.4)

$$A(t', \alpha') = \sum_l \left(\frac{2l+1}{4\pi}\right)^{1/2} B^{(n)}_l P_l(\cos\alpha') h^{(2)}_l(n_{n+1}kt'). \tag{4.20}$$

5. Application to a single lens

The preceding formulae are now applied to a very simple problem, the single lens. The notation is that of the left lens of figures 1 and 2, simplified to

$$t_1 = -\infty, \qquad n_1 = n_3 = 1, \qquad n_2 = n, \qquad d_2 = d. \tag{5.1}$$

Then (4.1), (4.2) give the usual equations

$$t'_1 = \frac{r}{n-1} \tag{5.2}$$

$$t_2 = \frac{r_1 - (n-1)d}{n-1} \tag{5.3}$$

$$t'_2 = \frac{n}{n-1}\,\frac{r_2[r_1-(n-1)d]}{r_2 - r_1 + (n-1)d}. \tag{5.4}$$

From (4.19) we get

$$\bar{B}_l^{(0)} = (8\pi l)^{1/2}(-\mathrm{i})^l. \tag{5.5}$$

Then (4.10) and (4.12) give

$$\bar{B}_{l'}^{(1)} = \begin{cases} -8\pi l(-\mathrm{i})^{l+1} nkh_l^{(1)}(kr_1)h^{(2)}(nkr_1) & l < \hat{l} \\ 0 & l > \hat{l}. \end{cases} \tag{5.6}$$

The value $\hat{l}$ is determined by (4.13):

$$\hat{\theta}_1 = \pi - \frac{1}{2}\left[\cos^{-1}\left(\frac{l}{kr_1}\right) + \frac{\pi}{2} + \cos^{-1}\left(\frac{l}{nkr_1}\right) + \cos^{-1}\left(\frac{(n-1)l}{nkr_1}\right)\right]. \tag{5.7}$$

Going now to the coordinate system at point O_2 we have from (4.14) and (4.15)

$$\bar{B}_{l'}^{(1)} = \sum_{l=0}^{\hat{l}} h_{|l'-l|}^{(2)}(nkd)B_l^{(1)}. \tag{5.8}$$

If there is no aperture limitation at surface σ_2, the refraction at σ_2 gives then simply

$$B_{l'}^{(2)} = \mathrm{i}kh_{l'}^{(1)}(nkr_2)h_{l'}^{(2)}(kr_2)B_{l'}^{(1)}. \tag{5.9}$$

Collecting together (5.6), (5.8) and (5.9), and using (4.20), we find the amplitude distribution of the image to be

$$A(t', \alpha') = 2nk^2 \sum_{l,l'} (ll')^{1/2} \mathrm{e}^{-\mathrm{i}\pi l/2} h_l^{(1)}(kr_1)h_l^{(2)}(nkr_1)h_{|l-l'|}^{(2)}(nkd)h_{l'}^{(1)}(nkr_2) \times h_{l'}^{(2)}(kr_2)h_{l'}^{(2)}(kt')P_{l'}(\cos\alpha'). \tag{5.10}$$

We have made a habit of dropping everywhere terms of the order l^{-1}.

Let us look at the amplitude distribution on the z axis, $\alpha' = 0$. Writing (5.10) as

$$A(t') = \sum_{l,l'} F_{ll'} \exp(\mathrm{i}\psi_{ll'}), \tag{5.11}$$

the amplitude and phase are obtained from the asymptotic expression

$$h_l^{(1)}(z) = z^{-1/2}(z^2 - l^2)^{-1/4} \exp\left\{\mathrm{i}\left[(z^2 - l^2)^{-1/2} - l\cos^{-1}\left(\frac{l}{z}\right) - \frac{\pi}{4}\right]\right\}. \tag{5.12}$$

In fact, they are

$$F_{ll'} = \frac{2(ll')^{1/2}}{k^4 n^2 r_1^2 d r_2^2 t'}\left[\left(1 - \frac{l^2}{k^2 r_1^2}\right)\left(1 - \frac{(l-l')^2}{n^2k^2d^2}\right)\left(1 - \frac{l'^2}{n^2k^2r_2^2}\right)\left(1 - \frac{l'^2}{k^2r_2^2}\right)\left(1 - \frac{l'^2}{k^2t'^2}\right)\right]^{-1/4} \tag{5.13}$$

$$\psi_{ll'} = -\tfrac{1}{2}\pi l + \Psi_l(kr_1) - \Psi_l(nkr_1) + \Psi_{|l-l'|}(nkd) + \Psi_{l'}(nkr_2) - \Psi_{l'}(kr_2) - \Psi_{l'}(kt') \tag{5.14}$$

where $\Psi_l(z)$ is defined as

$$\Psi_l(z) = (z^2 - l^2)^{1/2} - l\cos^{-1}\left(\frac{l}{z}\right) - \frac{\pi}{4}. \tag{5.15}$$

If the object point P_0 is at a finite distance t_1, then one must make the substitutions

$$1 \to \frac{1}{kt_1}\left(1-\frac{l^2}{k^2t_1^2}\right)^{-1/4} \tag{5.16}$$

in (5.13)

$$-\tfrac{1}{2}\pi l \to \Psi_l(kt_1) \tag{5.17}$$

in (5.14).

Let us study the change of phase under small changes of l and l'. The derivative is

$$d_l(z) = -\cos^{-1}\left(\frac{l}{z}\right) = \frac{\partial \Psi_l(z)}{\partial l}. \tag{5.18}$$

We also have

$$\frac{\partial}{\partial l}\Psi_{|l-l'|}(z) = -\frac{\partial}{\partial l'}\Psi_{|l-l'|}(z) = \pm d_{|l-l'|}(z). \tag{5.19}$$

At $l = 0, l' = 0$ both $\partial\Psi_{ll'}/\partial l$ and $\partial\Psi_{ll'}/\partial l'$ are zero. The sharpness of the image increases if the region of stationary phase in the l, l' plane becomes larger and larger. We then find the region where

$$\frac{\partial}{\partial l}\psi_{ll'} = -\tfrac{1}{2}\pi + d_l(kr_1) - d_l(nkr_1) + d_{|l-l'|}(nkd) \tag{5.20}$$

$$\frac{\partial}{\partial l'}\psi_{ll'} = \mp d_{|l-l'|}(nkd) + d_{l'}(nkr_2) - d_{l'}(kr_2) - d_{l'}(kt') \tag{5.21}$$

are smaller than some suitable constant, say $\tfrac{1}{2}\pi$.

For small x,

$$\cos^{-1} x = \tfrac{1}{2}\pi - x - \tfrac{1}{6}x^3 + \dots \tag{5.22}$$

implies

$$\frac{\partial}{\partial l}\psi_{ll'} = \frac{y}{r_1} - \frac{y}{nr_1} - \frac{y'-y}{nd} + \frac{1}{6}\left[\left(\frac{y}{r_1}\right)^3 - \left(\frac{y}{nr_1}\right)^3 - \left(\frac{y'-y}{nd}\right)^3\right] \tag{5.23}$$

$$\frac{\partial}{\partial l'}\psi_{ll'} = \frac{y'-y}{nd} + \frac{y'}{nr^2} - \frac{y'}{r^2} - \frac{y'}{t'} + \frac{1}{6}\left[\left(\frac{y'-y}{nd}\right)^3 + \left(\frac{y'}{nr_2}\right)^3 - \left(\frac{y'}{r_2}\right)^3 - \left(\frac{y'}{t'}\right)^3\right]. \tag{5.24}$$

We have introduced the variables

$$y = l/k, \qquad y' = l'/k. \tag{5.25}$$

One could now set some limits on the absolute values of (5.23) and (5.24) (or more accurately (5.20) and (5.21)), and keeping, eg, the equivalent focal length constant, find such r_1, r_2, d that these limits are not violated in as large a l, l' region as possible.

6. Solution for n surfaces

It is easy to see how (5.13) and (5.14) are generalized if there are n surfaces. In the notation of figures 1 and 2, the amplitude at the image is

$$A(t',\alpha') = \sum_{l_1=\check{l}_1}^{\hat{l}_1} \cdots \sum_{l_n=\check{l}_n}^{\hat{l}_n} F_{l_1\ldots l_n} \exp(\mathrm{i}\psi_{l_1\ldots l_n}) P_{l_n}(\cos\alpha') \tag{6.1}$$

$$\begin{aligned} F_{l_1\ldots l_n} = {} & \frac{2(l_1 l_n)^{1/2}}{t_1 t'}\left(1-\frac{l_1^2}{n_1^2k^2t_1^2}\right)^{-1/4}\left(1-\frac{l_n^2}{n_{n+1}^2k^2t'^2}\right)^{-1/4} \\ & \times \prod_{i=1}^{n}\left[(n_i n_{i+1}k^2r_i^2)^{-1}\left(1-\frac{l_i^2}{n_i^2k^2r_i^2}\right)^{-1/4}\left(1-\frac{l_i^2}{n_{i+1}^2k^2r_i^2}\right)^{-1/4}\right] \\ & \times \prod_{i=2}^{n}\left(1-\frac{(l_{i-1}-l_i)^2}{n_i^2k^2d_i^2}\right)^{-1/4} \end{aligned} \tag{6.2}$$

$$\begin{aligned} \psi_{l_1\ldots l_n} = {} & \Psi_{l_1}(n_1kt_1) + \sum_{i=1}^{n}(\Psi_{l_i}(n_ikr_i) - \Psi_{l_i}(n_{i+1}kr_i)) \\ & + \sum_{i=2}^{n}\Psi_{|l_{i-1}-l_i|}(n_ikd_i) - \Psi_{l_n}(n_{n+1}kt') \end{aligned} \tag{6.3}$$

where $\Psi_l(z)$ were defined in (5.15). The values $\check{l}_1,\ldots,\check{l}_n$ and $\hat{l}_1,\ldots,\hat{l}_n$ are solutions of (4.13). If t_i and t_i' satisfy (4.1), then $\check{l}_i = 0$.

In any practical situation the absolute value $F_{l_1\ldots l_n}$ varies so slowly that it can be regarded as constant. If t' has the value t_n' given by (4.1) for $i = n$, then the phase $\psi_{l_1\ldots l_n}$ is stationary at $l_1 = 0,\ldots, l_n = 0$. The intensity at the point $t' = t_n'$ is, in the first approximation, inversely proportional to the square of the determinant

$$\det\left(\frac{\partial^2}{\partial l_i\,\partial l_j}\right) \qquad i,j = 1,\ldots,n. \tag{6.4}$$

The amplitude can be calculated accurately by performing numerically the n fold integral (6.1). The upper limits $\hat{l}_i$ account for diffraction.

Appendix 1. Calculation of $\int \mathrm{d}t\, j_l(t)\, j_{l'}(t+u)$

To compute the integral over $j_l(t)j_{l'}(t+u)$ we substitute the asymptotic expression (A.13) of I:

$$\begin{aligned} & \int_{-\infty}^{\infty}\mathrm{d}t\; j_l(t+u)j_{l'}(t) \\ & \quad = \tfrac{1}{4}\int \mathrm{d}t\; t^{-1/2}(t+u)^{-1/2}(t^2-l^2)^{-1/4}[(t+u)^2-l'^2]^{-1/4} \\ & \qquad \times\exp\left\{\pm\mathrm{i}\left((t^2-l^2)^{1/2}-l\cos^{-1}\left(\frac{l}{t}\right)-\frac{\pi}{4}\right)\right. \\ & \qquad \left.\pm\mathrm{i}\left[[(t+u)^2-l'^2]^{1/2}-l'\cos^{-1}\left(\frac{l'}{t+u}\right)-\frac{\pi}{4}\right]\right\}, \end{aligned} \tag{A.1}$$

where a sum over the four sign combinations is implied. The integral (A.1) is evaluated by the method of stationary phase. The derivative of the phase gives

$$\pm\frac{(t^2-l^2)^{1/2}}{t}\pm\frac{[(t+u)^2-l'^2]^{1/2}}{t+u}=0 \tag{A.2}$$

which implies $l(t+u) = \pm l't$, and thus the phase is stationary at

$$t_0 = -\frac{l}{l\pm l'}u. \tag{A.3}$$

The sign combinations $+, +$ and $-, -$ in (A.1) require $+l'$ in (A.3), and the sign combinations $+, -$ and $-, +$ in (A.1) require $-l'$ in (A.3).

The second derivative of the phase at $t = t_0$ for the four sign combinations in (A.1) is

$$b=\frac{\partial^2\psi}{\partial t^2}=(\pm)\frac{(l+l')^4}{ll'u^2[u^2-(l+l')^2]^{1/2}}, \qquad \text{for } +, + \ (-, -) \tag{A.4}$$

$$b=\frac{\partial^2\psi}{\partial t^2}=(\pm)\frac{(l-l')^4}{ll'u^2[u^2-(l-l')^2]^{1/2}}, \qquad \text{for } +, - \ (-, +). \tag{A.5}$$

Now, making use of (A.9) in I and summing the four terms in (A.1) we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}t\, j_l(t+u)j_{l'}(t) &= \frac{(2\pi/ll')^{1/2}}{2[u^2-(l-l')^2]^{1/4}}\cos\left[[u^2-(l-l')^2]^{1/2}-u\cos^{-1}\left(\frac{l-l'}{u}\right)-\frac{\pi}{4}\right]\\ &\quad+\frac{(2\pi/ll')^{1/2}}{2[u^2-(l+l')^2]^{1/4}}\cos\left[[u^2-(l+l')^2]^{1/2}-u\cos^{-1}\left(\frac{l+l'}{u}\right)-\frac{\pi}{4}\right]\end{aligned} \tag{A.6}$$

which is

$$\int_{-\infty}^{\infty} \mathrm{d}t\, j_l(t+u)j_{l'}(t) = \frac{\pi}{2(ll')^{1/2}}(j_{|l-l'|}(u)+j_{l+l'}(u)). \tag{A.7}$$

The formulae

$$\int_{-\infty}^{\infty} \mathrm{d}t\, h_l^{(2)}(t)h_{l'}^{(2)}(t+u) = \frac{\pi}{(ll')^{1/2}}h_{l+l'}^{(2)}(u) \tag{A.8}$$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, h_l^{(1)}(t)h_{l'}^{(2)}(t+u) = \frac{\pi}{(ll')^{1/2}}h_{|l'-l|}^{(2)}(u) \tag{A.9}$$

$$\int_{-\infty}^{\infty} \mathrm{d}t\, j_l(t)h_{l'}^{(2)}(t+u) = \frac{\pi}{2(ll')^{1/2}}(h_{|l'-l|}^{(2}(u)+h_{l'+l}^{(2)}(u)) \tag{A.10}$$

can be derived in the same way.

References

Byckling E 1974 *J. Phys.* A: *Math., Nucl. Gen.* **7** 505–18

van Bladel J 1964 *Electromagnetic Fields* (New York: McGraw-Hill)